

A simplified version of the ‘Axis of Evil Theorem’ for distinct points.

Michela Ceria
Università degli Studi di Torino.
michela.ceria@unito.it

Abstract

Given a finite set \mathbf{X} of distinct points, Marinari-Mora’s ‘Axis of Evil Theorem’ states that a combinatorial algorithm and interpolation enable to find a ‘linear’ factorization for a lexicographical minimal Groebner basis $\mathcal{G}(I(\mathbf{X}))$ of the zerodimensional radical ideal $I(\mathbf{X})$.

In this work we provide such algorithm, showing that it ends in a finite number of steps and that it actually provides the correct result.

The ‘Axis of Evil’ algorithm takes as input the monomial basis of the initial ideal $T(I(\mathbf{X}))$ but its starting point is the (finite) Groebner escalier N (obtained via Cerlienco-Mureddu correspondence) so we will also define the ‘potential expansion’ algorithm, a combinatorial algorithm which computes the minimal basis from a finite Groebner escalier.

Keywords: Groebner basis , Combinatorial algorithm, Interpolation.

1 Introduction.

Marinari-Mora in [10], [9], [11] gave a deep description of the structure of a zero-dimensional ideal I described by giving its Macaulay basis $\mathcal{B}(I)$ ([16]); in particular they enhanced the description of the Grobner basis of an ideal in $K[X, Y]$ given by Lazard in [8] proving that in a restricted case which includes the radical one, for each monomial $\tau := X_1^{d_1} \cdots X_n^{d_n}$ belonging to the minimal basis $G(I)$ of the initial ideal of I , it is possible to produce linear factors $\gamma_{m\delta\tau} := X_i - f(X_1, \dots, X_{i-1})$, $1 \leq m \leq n, 1 \leq \delta \leq d_m$ such that the polynomials $f_\tau := \prod_{m=1}^n \prod_{\delta=1}^{d_m} \gamma_{m\delta\tau}$ form a minimal lexicographical Groebner basis of I ; each such factors were obtained by producing an appropriate decomposition of the given Macaulay basis $\mathcal{B}(I) = \bigsqcup_{m=1}^n \bigsqcup_{\delta=1}^{d_m} S_{m\delta}(\tau)$ and interpolating over the monomial set obtained applying Cerlineco-Mureddu Algorithm over the set of functionals $S_{m\delta}(\tau)$.

Such algorithm is reported and proved in [16]; later Mora in a series of lecture notes labelled the restriction of this decomposition and interpolation to the case of a set of distinct points as ‘Axis-of-Evil’ theorem and gave a precise description, but no simple proof, of the result stated in [16]; S. Steidel implemented the procedure in Singular [6], [18].

We give here such explicit algorithm that, given a finite set \mathbf{X} of distinct points, provides a complete decomposition $\mathbf{X} = \bigsqcup_{m=1}^n \bigsqcup_{\delta=1}^{d_m} S_{m\delta}(\tau)$ on which, applying Cerlienco-Mureddu algorithm and interpolation, produces the required linear

factorization for a lexicographical minimal Groebner basis $F = \{f_1, \dots, f_r\}$ of the ideal $I(\mathbf{X})$ and thus a very simple proof of the ‘Axis-of-Evil’ theorem in this particular situation.

This algorithm arranges the r terms t_i belonging to $G(I(\mathbf{X}))$ with respect to lex ($t_1 \leq \dots \leq t_r$) and constructs the factorization of each $f_i \in F$ through a suitable interpolation on a subset $S_{m\delta}(t_i)$ of \mathbf{X} depending on the exponents of the corresponding t_i . More precisely, Cerlienco-Mureddu give an algorithm that enables to find the Groebner escalier $N(I(\mathbf{X}))$ and the minimal basis $G(I(\mathbf{X}))$ of the monomial ideal $T(I(\mathbf{X}))$.

Since the ‘Axis of Evil’ algorithm’s starting point are the elements of \mathbf{X} and the monomials of the finite Groebner escalier N (computed using Cerlienco-Mureddu algorithm), but the algorithm requires as input the monomial basis of $T(I(\mathbf{X}))$, we also define the ‘potential expansion’ algorithm.

It takes N and computes the minimal basis.

I note here that Marinari-Mora explicitly deduced, as trivial corollaries of their ‘Axis-of-Evil’ procedure, Lazard theorem ([8]), Elimination theorem ([2]), Kalkbrenner theorem ([13]), part of Gianni-Kalkbrenner theorem ([7],[12]); they however remarked that, having being strongly influenced by Gianni-Kalkbrenner result, they cannot dismiss the possibility that Gianni-Kalkbrenner argument is an essential tool of their proof of the ‘Axis-of-Evil’ theorem.

2 Notation.

Let $P := k[x_1, \dots, x_n] = \bigoplus_{d \in \mathbb{N}} P_d$ be the ring of polynomials in n variables and coefficients in the base field k . For all $M \subseteq P$, $M_d = M \cap P_d$ is its degree d part. Call \mathcal{T} the semigroup of terms, generated by the set $\{x_1, \dots, x_n\}$:

$$\mathcal{T} := \{x_1^{a_1} \cdots x_n^{a_n}, (a_1, \dots, a_n) \in \mathbb{N}^n\}.$$

Letting $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will often write x^α instead of $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Define also the set

$$T[m] := \mathcal{T} \cap k[x_1, \dots, x_m] = \{x_1^{a_1} \cdots x_m^{a_m} / (a_1, \dots, a_m) \in \mathbb{N}^m\}.$$

For each *semigroup ordering* $<$ on \mathcal{T} (i.e. a total ordering such that $t_1 < t_2 \Rightarrow tt_1 < tt_2, \forall t, t_1, t_2 \in \mathcal{T}$) we can represent a polynomial $f \in P$ as a linear combination (with coefficients in k) of monomials arranged w.r.t. $<$:

$$f = \sum_{t \in \mathcal{T}} c(f, t)t = \sum_{i=1}^s c(f, t_i)t_i : c(f, t_i) \in k^*, t_i \in \mathcal{T}, t_1 > \dots > t_s.$$

We will call $T(f) = Lt(f) := t_1$ the *leading term* of f and $tail(f) = f - T(f)$ the *tail* of f .

We can also express it in a unique way as

$$f = \sum_{i=0}^{\delta} g_i x_n^i \in k[x_1, \dots, x_{n-1}][x_n], g_i \in k[x_1, \dots, x_{n-1}], g_\delta \neq 0$$

(where $\delta := \deg_n(f)$ is the degree w.r.t. x_n).

We denote $Lp(f) := g_\delta$, the *leading polynomial* of f .

Definition 2.1. For each monomial $t \in \mathcal{T}$ and $x_j|t$, the only $u \in \mathcal{T}$ such that $t = x_j u$ is called j -th predecessor of t .

A subset $N \subseteq \mathcal{T}$ is an *order ideal* if

$$t \in N \Rightarrow s \in N \forall s|t.$$

Let $N \subset \mathcal{T}$ an order ideal, A subset $N \subseteq \mathcal{T}$ is an order ideal if and only if $\mathcal{T} \setminus N = J$ is a semigroup ideal (i.e. $\tau \in J \Rightarrow t\tau \in J, \forall t \in \mathcal{T}$).

We set $N(J) := \mathcal{T} \setminus T(J) = N$.

For a semigroup ideal J , $G(J)$ denotes its minimal basis and

$$\begin{aligned} G(J) &:= \{\tau \in J \mid \text{each predecessor of } \tau \in N(J)\} = \\ &= \{\tau \in \mathcal{T} \mid N(J) \cup \{\tau\} \text{ order ideal, } \tau \notin N(J)\}. \end{aligned}$$

For all subsets $G \subset P$, we define $T\{G\} := \{T(g), g \in G\}$ and we call $T(G)$ the semigroup ideal $\{\tau T(g), \tau \in \mathcal{T}, g \in G\}$, generated by $T\{G\}$.

For any ideal $I \triangleleft P$ consider the semigroup ideal $T(I) = T\{I\}$, denoting by abuse of notation $G(I)$ its minimal basis $G(I)$ and the border ideal of I

$$\begin{aligned} B(I) &:= \{x_h t, 1 \leq h \leq n, t \in N(I)\} \setminus N(I) = \\ &= T(I) \cap (\{1\} \cup \{x_h t, 1 \leq h \leq n, t \in N(I)\}). \end{aligned}$$

We will always consider the *lexicographic order* induced by $x_1 < \dots < x_n$, i.e:

$$x_1^{a_1} \dots x_n^{a_n} < x_1^{b_1} \dots x_n^{b_n} \Leftrightarrow \exists j \mid a_j < b_j, a_i = b_i, \forall i > j.$$

This is a *term order*, that is a semigroup ordering such that 1 lower to every variable or, equivalently, it is a *well ordering*.

Lemma / Definition 2.2. We have:

1. $P \cong I \oplus k[N(I)];$
2. $P/I \cong k[N(I)];$
3. $\forall f \in P, \exists! g := \text{Can}(f, I) = \sum_{t \in N(I)} \gamma(f, t, <) t \in k[N(I)],$ called canonical form of f with respect to I , such that $f - g \in I$.

Definition 2.3. Given a term order $<$ on \mathcal{T} :

1. a Groebner basis of I is a set $G \subset I$ such that $T(G) = T\{I\}$, that is $T\{G\}$ generates the semigroup ideal $T(I) = T\{I\}$;
2. a minimal Groebner basis is a Groebner basis such that divisibility relations among the leading monomials of its members do not exist;
3. the unique reduced Groebner basis of I is the set:

$$\mathcal{G}(I) := \{\tau - \text{Can}(\tau, I) : \tau \in G(I)\}.$$

Each member of the reduced Groebner basis has a leading term which does not divide any monomial of another member.

Let $\mathbf{X} = \{P_1, \dots, P_N\} \subset k^n$ be a finite set of distinct points

$$P_i := (a_{i1}, \dots, a_{in}), i = 1, \dots, N.$$

We call

$$I(\mathbf{X}) := \{f \in P : f(P_i) = 0, \forall i\},$$

the *ideal of points* of \mathbf{X} .

Finally, we define the projection maps:

$$\pi_m : k^n \rightarrow k^m \qquad \pi^m : k^n \rightarrow k^{n-m+1}$$

$$(X_1, \dots, X_n) \mapsto (X_1, \dots, X_m), \qquad (X_1, \dots, X_n) \mapsto (X_m, \dots, X_n)$$

and, for $P \in k^n$, $\mathbf{X} \subset k^n$, let

$$\Pi_s(P, \mathbf{X}) := \{P_i \in \mathbf{X} / \pi_s(P_i) = \pi_s(P)\},$$

$$\Pi^s(P, \mathbf{X}) := \{P_i \in \mathbf{X} / \pi^s(P_i) = \pi^s(P)\},$$

extending in the obvious way the meanings of $\pi_s(\mathbf{d}), \pi^s(\mathbf{d}), \Pi_s(\mathbf{d}, D), \Pi^s(\mathbf{d}, D)$ to $\mathbf{d} \in \mathbb{N}^n \subset k^n$ e $D \subset \mathbb{N}^n \subseteq \mathbb{N}^n$.

With the same notation π_m we indicate also

$$\pi_m : \mathcal{T} \cong \mathbb{N}^n \rightarrow \mathbb{N}^m \cong T[m]$$

$$x_1^{a_1} \dots x_n^{a_n} \mapsto x_1^{a_1} \dots x_m^{a_m}.$$

3 Cerlienco-Mureddu Correspondence.

Cerlienco and Mureddu ([3], [4], [5]) provided an algorithm which solves the following

Problem: Given finite ordered set of distinct points

$$\underline{\mathbf{X}} := (P_1, \dots, P_N) \subset k^n; P_i := (a_{i1}, \dots, a_{in})$$

compute a monomial basis (w.r.t. the lexicographic order induced by $x_1 < \dots < x_n$) of the quotient $k[x_1, \dots, x_n]/I(\mathbf{X})$, where \mathbf{X} denotes the support $\{P_1, \dots, P_N\}$ of $\underline{\mathbf{X}}$. \circledast

More precisely, they

- define the operator Φ , associating to $\underline{\mathbf{X}}$ an ordered set

$$\Phi(\underline{\mathbf{X}}) := (\mathbf{d}_1, \dots, \mathbf{d}_N) \subset \mathbb{N}^n$$

such that $|\Phi(\underline{\mathbf{X}})| = |\underline{\mathbf{X}}| = N$ and such that, for all $m < N$ the subset $(\mathbf{d}_1, \dots, \mathbf{d}_m)$ is exactly $\Phi((P_1, \dots, P_m))$.

- define the σ -value w.r.t. \mathbf{X} $s = \sigma(P, \mathbf{X})$ of a point $P \in K^n \setminus \mathbf{X}$ as the maximal integer such that $\Pi_{s-1}(P, \mathbf{X}) \neq \emptyset$ (by convention, $\forall P, \mathbf{X}$, $\Pi_0(P, \mathbf{X}) \neq \emptyset$).

For $P \notin \mathbf{X}$, they define the set

$$\Sigma(P, \mathbf{X}) := \{P_i \in \mathbf{X} / \pi_{s-1}(P_i) = \pi_{s-1}(P), s = \sigma(P, \mathbf{X})\}$$

containing all the points of \mathbf{X} having the first $s - 1$ coordinates equal to those of $P \notin \mathbf{X}$. They extend the notation to the case $P = P_j \in \underline{\mathbf{X}}$ in the following way:

$$\sigma(P, \underline{\mathbf{X}}) := \sigma(P, \{P_1, \dots, P_{j-1}\})$$

$$\Sigma(P, \underline{\mathbf{X}}) := \Sigma(P, \{P_1, \dots, P_{j-1}\}).$$

Remark 3.1. Given a term order \preceq , a monomial basis for $A := k[x_1, \dots, x_n]/I(\mathbf{X})$,

$$[\mathbf{x}^{i_1}], \dots, [\mathbf{x}^{i_N}], \text{ with } \mathbf{x}^{i_1} \preceq \dots \preceq \mathbf{x}^{i_N}$$

is called minimal with respect to the term order if, for every other monomial basis $[\mathbf{x}^{i'_1}], \dots, [\mathbf{x}^{i'_N}]$, with $\mathbf{x}^{i'_1} \preceq \dots \preceq \mathbf{x}^{i'_N}$ for the A it holds

$$\forall j = 1, \dots, N, \mathbf{x}^{i_j} \preceq \mathbf{x}^{i'_j}.$$

In [3], they state that the computed monomial basis is the minimal one.

Proposition 3.2. ([3])

Let $D := \Phi(\mathbf{X})$. Then $\{[\mathbf{x}^{\mathbf{d}}]/\mathbf{d} \in D\}$ is a monomial basis for $k[x_1, \dots, x_n]/I(\mathbf{X})$. Such a monomial basis is minimal with respect to the given $<$.

Once the Groebner escalier N is known, it is very simple to compute the minimal basis G of $T(I(\mathbf{X})) = \mathcal{T} \setminus N$. Given the set \mathbf{X} , the first step to compute the linear factorization of a minimal Groebner basis will be to apply Cerlienco-Mureddu algorithm to \mathbf{X} and compute N , in order to obtain G .

4 The potential expansion's algorithm.

Consider the polynomial ring $k[x_1, \dots, x_n]$ with usual ordering $<$. Given a finite set of distinct points $\mathbf{X} = \{P_1, \dots, P_N\}$, consider the ideal $I(\mathbf{X}) \triangleleft k[x_1, \dots, x_n]$ which is radical and zerodimensional, so its Groebner escalier N is a finite set. We will compute the minimal monomial basis G of $T(I(\mathbf{X}))$, given the Groebner escalier. The algorithm actually provides correct results irrespective of the given term ordering, but since we use Cerlienco-Mureddu correspondence, we will consider only our lex order.

In order to make the reasoning clear, we will represent the monomials using the same diagrams introduced in [15] to study properties of Borel ideals.

Apply Cerlienco-Mureddu correspondence to \mathbf{X} in order to have $N(\mathbf{X}) = \{\tau_1, \dots, \tau_N\}$. It is well known (see, for instance [16]) that $|N(\mathbf{X})| = |\mathbf{X}|$.

We first define the potential expansion of a subset $H \subset \mathcal{T}$, from which the algorithm bears its name.

Definition 4.1. Let $H \subseteq \mathcal{T}_j$ for some $j \in \mathbb{N}^*$ we set $C^{(0)}(H) := H$ and, for all $l \in \mathbb{N}^*$ $C^{(l)}(\tau) = \mathcal{T}_{j+l} \setminus \{x_1, \dots, x_n\} \cdot (\mathcal{T}_{j+l-1} \setminus C^{(l-1)}(H))$.

We then slice the Groebner escalier by degree, having N_0, N_1, \dots, N_h , where h is the maximal degree of terms appearing in N .

The minimal monomial basis $G(I(\mathbf{X}))$ will have at most degree $h + 1$. As a matter of fact, if $\tau \in G$ with $\deg(\tau) = d > h + 1$ its predecessors will belong to N and have degree $d - 1 \geq h + 1$ which is impossible.

Algorithm 1 Cerlienco-Mureddu algorithm.

```

1: procedure CEMU( $\underline{\mathbf{X}}$ )  $\rightarrow \Phi(\underline{\mathbf{X}})$ 
2:   if  $N = 1$  then
3:      $\Phi(\underline{\mathbf{X}}) := \{(0, \dots, 0)\}$ .
4:   end if
5:   if  $1 < N$  then  $\triangleright$  suppose to know by induction hypothesis  $\Phi((P_1, \dots, P_{N-1})) = (\mathbf{d}_1, \dots, \mathbf{d}_{N-1})$ 
     and look for  $\mathbf{d}_N = \Phi(P_N)$ .
6:      $s = \sigma(P_N, \underline{\mathbf{X}})$ .
7:     for  $i = n$  to 1 do
8:       if  $i > s$  then
9:          $d_{Ni} = 0$ .
10:      end if
11:      if  $i = s$  then
12:         $m, (1 \leq m \leq n)$ , maximal s.t.  $\pi_{s-1}(P_m) = \pi_{s-1}(P_N)$ ,
         $\pi^{s+1}(\mathbf{d}_m) = (0, \dots, 0) = \pi^{s+1}(\mathbf{d}_N)$ .  $\triangleright P_m$  is the  $\sigma$ -antecedent of  $P_N$  w.r.t.  $(P_1, \dots, P_{N-1})$ ,
         $\Phi((P_1, \dots, P_{N-1}))$ .
13:         $d_{Ns} = d_{ms} + 1$ .
14:      end if
15:      if  $i < s$  then  $\triangleright$  we use induction here.
16:         $\mathcal{W}(P_N, \underline{\mathbf{X}}) := \{P \in \underline{\mathbf{X}} \mid \Phi(P) = \mathbf{d} = (*, \dots, *, d_{Ns}, 0, \dots, 0), \} =$ 
 $\{P_{j1}, \dots, P_{jr}\}$ .
17:         $\mathcal{Q} := \pi_{s-1}(\mathcal{W}(P_N, \underline{\mathbf{X}}))$ .  $\triangleright$ 
 $|\mathcal{Q}| = |\mathcal{W}(P_N, \underline{\mathbf{X}})| = r < N$ . If  $h < r = |\mathcal{W}(P_N, \underline{\mathbf{X}})|$ , then  $\pi_{s-1}(P_{jh}) \neq \pi_{s-1}(P_N)$ . Moreover, since  $\Phi$  is
        inductive, if  $h < k \leq r$  then  $\pi_{s-1}(P_{jh}) \neq \pi_{s-1}(P_{jk})$ .
18:         $\pi_{s-1}(\mathbf{d}_N) = \widetilde{\mathbf{d}_r}$ .  $\triangleright$  By the induction hypothesis,  $\Phi(\underline{\mathcal{Q}}) = (\widetilde{\mathbf{d}}_1, \dots, \widetilde{\mathbf{d}}_r)$  and
         $\forall 1 \leq i < r, \widetilde{\mathbf{d}}_i = \pi_{s-1}(\mathbf{d}_{ji})$ .
19:        break.
20:      end if
21:    end for
22:  end if
23:  return  $\Phi(\underline{\mathbf{X}})$ .
24: end procedure

```

The computation of G is performed as follows.

Consider $\mathcal{T}_i \forall i = 0, \dots, h+1$: it is well known that $|\mathcal{T}_i| = \binom{n+i-1}{n-1}$.

For each i , define $Gen_i(I) := \{t \in G(I) \mid \deg(t) \leq i\}$. Since I is a proper ideal, $Gen_0(I) = \emptyset$.

Let h the minimal i such that $Gen_h(I) \neq \emptyset, \forall i \geq 1$

$$Gen_{i+h} = Gen_{h+i-1} \cup (\mathcal{T}_{h+i} \setminus (N_{h+i} \cup \bigcup_{j=h+1}^{h+i-1} C^{(h+i-j)}(G_j))).$$

We then have

Algorithm 2 The potential expansion's algorithm.

```

1: procedure POTEXP( $N(I) \rightarrow I$ ) ▷  $I$  is expressed using its minimal basis.
Require:  $N = [N_0, \dots, N_h, N_{h+1}]$ , such that  $N_{h+1} = \emptyset$ .
2:    $C = [\emptyset]$ . ▷ the potential expansion's list.
3:    $Gen = \emptyset$ .
4:    $I = (0)$ .
5:   for  $i = 0$  to  $h+1$  do
6:      $d = \binom{n+\deg(N_i[1])-1}{n-1} - |N_i \cup C[i]|$ .
7:     if  $d = 0$  then ▷ no new generators.
8:        $C[i+1] = PotentialExpansion(C[i])$ .
9:        $Gen_i = (0)$ 
10:    else ▷ adding new generators.
11:       $Gen_i = \mathcal{T}_i \setminus (N_i \cup C[i])$ .
12:       $C[i+1] = PotentialExpansion(Gen_i \cup C[i])$ .
13:       $I = I + Gen_i$ .
14:    end if
15:  end for
16: return  $I$ 
17: end procedure

```

The algorithm uses a subroutine *PotentialExpansion* such that

$$PotentialExpansion(F) = C^{(1)}(F).$$

We will also have a subroutine finding $\mathcal{T}_{h+i} \setminus (N_{h+i} \cup \bigcup_{j=h+1}^{h+i-1} C^{(h+i-j)}(G_j))$.

WLOG we will think that the sets \mathcal{T}_{h+i} and $N_{h+i} \cup \bigcup_{j=h+1}^{h+i-1} C^{(h+i-j)}(G_j)$ are ordered w.r.t. the same ordering, since it is enough to perform a merging with the Groebner escalier and the potential expansion previously ordered.

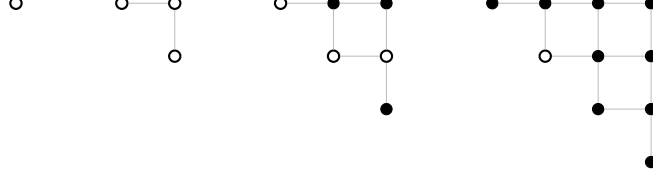
All these steps end: the subroutine finding the complementary can be developed performing a loop on the two ordered lists $A := \mathcal{T}_i = [a_1, \dots, a_m]$, $m \geq n$ and $B := N_i \cup C^{(i)} = [b_1, \dots, b_n]$ (using two indices i, j), keeping in mind that $B \subseteq A$ or $B = A$ and that $B[j] \geq A[i]$ at every step. Start with b_1 : if $b_1 = a_1$ we set $i = i+1; j = j+1$.

If we find $a_i \neq b_j$ for a certain couple (i, j) , we put $A[i]$ in the complementary and $i = i+1$ without modifying j .

Example 4.2. *There are situations in which N contains monomials of degree at most h , but also the minimal basis shares the same property.*

Take $I = (x^3, y^2, z^2, xy) \triangleleft k[x, y, z]$, whose Groebner escalier is:

$$\begin{aligned}
N_0 &= \{1\} \\
N_1 &= \{x, y, z\} \\
N_2 &= \{yz, xz, x^2\} \\
N_3 &= \{x^2z\}:
\end{aligned}$$



The monomial basis does not contain elements of degree 4.

We call G_i the set of i -degree elements of the minimal basis and I the monomial ideal we want to find.

Lemma 4.3. For all $i = 0, \dots, h+1$

$$T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j)) = G_i.$$

Proof: The inclusion $T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j)) \supseteq G_i$ is trivial, so we only prove $T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j)) \subseteq G_i$.

Consider $\tau \in T_i \setminus (N_i \cup \bigcup_{j=1}^{i-1} C^{i-j}(G_j))$. Clearly $\tau \in I$.

Let σ the i th predecessor of τ ; if $\sigma \in I$, $\exists t \in G$ with $\sigma = t \cdot m$ for a suitable $m \in \mathcal{T}$.

Then $\tau = t \cdot m \cdot x_i$ i.e. $\tau \in \bigcup_{j=1}^{i-1} C^{i-j}(G_j)$. \diamond

This lemma assures that the result obtained via the potential expansion's algorithm is correct.

5 The Axis of Evil Algorithm.

A 0 -dimensional radical ideal $I \triangleleft P$ is completely determined if we know the set $V(I)$ of its zeros.

Consider a finite set of distinct points $\mathbf{X} = \{P_1, \dots, P_r\}$; we will denote indifferently the Groebner escalier of the ideal $I(\mathbf{X})$ with $N(I(\mathbf{X}))$ or N . A variation of Cerlienco-Mureddu algorithm ([3]) allows us to find a 'linear factorization' for every element of a lexicographic minimal Groebner basis in the sense of the

Theorem 5.1. Let $t_i := x_1^{d_1} \dots x_n^{d_n}$, $i = 1, \dots, r$ be the generators of the minimal basis of $T(I)$, where I is a 0 -dimensional radical ideal.

A combinatorial algorithm and interpolation allow us to deduce polynomials

$$\gamma_{m\delta i} = x_m - g_{m\delta i}(x_1, \dots, x_{m-1}),$$

$\forall i, m, \delta$, with $1 \leq i \leq r$, $1 \leq m \leq n$, $1 \leq \delta \leq d_m$ such that

$$f_i = \prod_m \prod_{\delta} \gamma_{m\delta i} \quad \forall i$$

where $f_i, i = 1, \dots, r$ are the polynomials forming a minimal Groebner basis of I with respect to the lexicographic order induced by $x_1 < \dots < x_n$.

In that algorithm we will use the projections, as we defined in section 3. The Axis of Evil algorithm works then in the following way:

- consider $\tau_j := x_1^{d_1} \dots x_n^{d_n} \in G$. The required polynomial $f = \tau_j + \text{tail}(f)$ is factorized in $\sum_{i=1}^n d_i$ factors: d_1 polynomials whose leading term is x_1 , d_2 polynomials such that their leading term is x_2 and so on;
- consider the monomials $x_1^{a_1} x_2^{d_2} \dots x_n^{d_n}$ such that $a_1 < d_1$;
- every such monomial is associated, via Cerlienco-Mureddu Correspondence, to a point of our set \mathbf{X} . Project these points with respect to the first coordinate, obtaining d_1 numbers y_1, \dots, y_{d_1} ;
- $x_1 - y_i, i = 1, \dots, d_1$ are the first d_1 factors;
- construct the subset D_{20} of \mathbf{X} containing all the points in which the product $(x_1 - y_1) \dots (x_1 - y_{d_1})$ does not vanish. If it is empty then stop and consider the next monomial in G ; otherwise continue as follows;
- find the set $N_2(\tau_j)$ of all monomials in $\mathcal{T}[2]$ such that $x_1^{\alpha_1} x_2^{\alpha_2} < x_1^{d_1} x_2^{d_2}$;
- split the elements of $N_2(\tau_j)$ with respect to the exponents of x_2 and construct, via Cerlienco-Mureddu correspondence, the set

$$\{\Phi^{-1}(v x_2^{d_2-\delta} x_3^{d_3} \dots x_n^{d_n}) / v \in T[1], v x_2^{d_2-\delta} \in N_2(\tau_j)\}$$

- intersect the previous set with D_{20} , project the resulting set of points ($A_{2\delta}(\tau_j)$) with respect to the first two coordinates and apply Cerlienco-Mureddu Correspondence, obtaining a set $E_{2\delta\tau_j}$;
- interpolate over $A_{2\delta}(\tau_j)$, finding d_2 factors whose leading terms are all equal to x_2 . The monomials of $E_{2\delta\tau_j}$ are the ones appearing in such factorization;
- update the set of points in which the current polynomial does not vanish and stop if it is empty;
- repeat these steps letting all the variables vary one by one;
- repeat all the steps for all $\tau_i \in G$.

Remark 5.2. Given $\tau_j = x_1^{d_1} \dots x_n^{d_n} \in G$, every variable x_i will appear only d_i times in the execution of the algorithm.

Remark 5.3. The sets $N_m(\tau_j) := \{\omega \in T[m], \tau_j > \omega x_{m+1}^{d_{m+1}} \dots x_n^{d_n} \in N\}$ (in particular for $m = 1$ we have $N_1(\tau_j) := \{x_1^i / i < d_1\}$) are constructed in order to determine in which points it is necessary to interpolate.

Since for $\mu > \tau_j$ the Cerlienco-Mureddu correspondence provides a point $P_{\mu'}$ such that $\exists k \in \{1, \dots, n\} : \pi_k(P_{\mu}) = \pi_k(P_{\mu'})$, in order to obtain polynomials vanishing on all the point of \mathbf{X} it is not necessary to interpolate in the whole $\Phi^{-1}(N)$ as it suffices to consider only those corresponding to $\mu \in N$ with $\mu < \tau_j$.

Algorithm 3 The Axis of Evil algorithm.

```

1: procedure AoE( $\mathbf{X}, G(I(\mathbf{X})) := \{\tau_1, \dots, \tau_r\} \rightarrow R \triangleright R$  contains a factorized minimal
   Groebner basis of  $I$ .
Require: the elements  $G(I(\mathbf{X}))$  are in increasing order w.r.t the lexicographical
   order w.r.t.  $x_1 < \dots < x_r$ .
2:    $R = \emptyset$ 
3:   for  $i = 1$  to  $r$  do
4:      $N_1(\tau_j) := \{x_1^i / i < d_1\} = \{\omega \in T[1], \tau_j > \omega x_2^{d_2} \dots x_n^{d_n} \in N\}$ 
5:      $A_1(\tau_j) := \{\Phi^{-1}(x_1^i x_2^{d_2} \dots x_n^{d_n}) / i < d_i\} \subset \mathbf{X}$ .
6:      $B_1(\tau_j) := \pi_1(A_1(\tau_j)) \subset k$ .
7:      $\gamma_{1\tau_j} := \prod_{a \in B_1(\tau_j)} (x_1 - a)$ .
8:     for  $m = 2$  to  $n$  do
9:        $\zeta_{m\tau_j} := \prod_{\nu=1}^{m-1} \gamma_{\nu\tau_j}$ .
10:       $D_{m0} := \{P_i \in \mathbf{X} / \zeta_{m\tau_j}(P_i) \neq 0\}$ .
11:      if  $|D_{m0}| = 0$  then
12:         $R = [R, \zeta_{m\tau_j}]$ .
13:        break.
14:      end if
15:       $N_m(\tau_j) := \{\omega \in T[m], \tau_j > \omega x_{m+1}^{d_{m+1}} \dots x_n^{d_n} \in N\}$ .
16:      for  $\delta = 1$  to  $d_m$  do
17:         $A_{m\delta}(\tau_j) := \{\Phi^{-1}(v x_m^{d_m-\delta} x_{m+1}^{d_{m+1}} \dots x_n^{d_n}) / v \in T[m -$ 
18:         $1], v x_m^{d_m-\delta} \in N_m(\tau_j)\} \cap D_{m(\delta-1)}(\tau_j)$ .
19:         $E_{m\delta}(\tau_j) := \Phi(\pi_m(A_{m\delta}(\tau_j)))$ .
20:         $\gamma_{m\delta\tau_j} := x_m + \sum_{\omega \in E_{m\delta}(\tau_j)} c(\gamma_{m\tau_j}, \omega)\omega,$ 
21:        such that  $\gamma_{m\delta\tau_j}(P) = 0, \forall P \in A_{m\delta}(\tau_j)$ .
22:         $\xi_{m\delta} := \prod_{\nu=1}^{m-1} \gamma_{\nu\tau_j} \prod_{d=1}^{\delta} \gamma_{m d \tau_j}$ .
23:         $D_{m\delta}(\tau_j) := \{P_i \in \mathbf{X} / \xi_{m\delta}(P_i) \neq 0\} \subseteq \mathbf{X}$ 
24:        if  $|D_{m\delta}(\tau_j)| = 0$  then
25:           $R = [R, \xi_{m\delta}]$ .
26:          break.
27:        end if
28:      end for
29:       $\gamma_{m\tau_j} := \prod_{\delta} \gamma_{m\delta\tau_j}$ .
30:    end for
31:  return  $R$ .
end procedure

```

Remark 5.4. The terms smaller than τ_j mentioned before are found releasing all the variables one by one.

Imagine the monomials in $k[x_1, \dots, x_n]$ as points in k^n , identifying every term to the n -uple of its exponents. So we can ‘draw’ them in a n -dimensional space and we can think our releasings as an increment by one of the ‘directions’ where we can move there.

We point out that $N_m(\tau_j) \subseteq N_h(\tau_j)$ for $m \leq h$.

If $\omega \in N_m(\tau_j)$, $\tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N$; as $\omega x_{h+1}^{d_{h+1}} \cdots x_n^{d_n} | x_{m+1}^{d_{m+1}} \cdots x_n^{d_n}$ we have $\omega x_{h+1}^{d_{h+1}} \cdots x_n^{d_n} \in N$ and

$$\omega x_{h+1}^{d_{h+1}} \cdots x_n^{d_n} \leq x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} < \tau_j.$$

At each step we will count out all the points in which the polynomial already vanishes and we will stop the computation when the current factorized polynomial vanishes on the whole \mathbf{X} .

We will see an example of it later.

Remark 5.5. If the number of released variables is > 1 , we also must split the obtained monomials regarding the exponent of the maximal variable.

Consider then the loop on δ and, in particular, the set:

$$C_{m\delta}(\tau_j) := \{\Phi^{-1}(v x_m^{d_m-\delta} x_{m+1}^{d_{m+1}} \cdots x_n^{d_n}) / v \in T[m-1], v x_m^{d_m-\delta} \in N_m(\tau_j)\}.$$

We intersect $C_{m\delta}(\tau_j)$ with the subset of \mathbf{X} containing the points not vanishing the current factorized polynomial.

We can easily notice that, performing the algorithm, we only compute the sets $C_{m1}(\tau_j), \dots, C_{md_m}(\tau_j)$, but in $N_m(\tau_j)$ there are also monomials $\omega = x_1^{a_1} \cdots x_{m-1}^{a_{m-1}} x_m^{d_m}$ such that $\tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N$, which would be generated considering $\delta = 0$. They are not considered in the algorithm because they are related to monomials examined in the previous step: $= x_1^{a_1} \cdots x_{m-1}^{a_{m-1}} \in N_{m-1}$, so the corresponding points have already been treated. Taking $\delta = 0, \dots, d_m$, the sets $C_{m\delta}(\tau_j)$ form a partition of $N_m(\tau_j)$ basing on the degree of x_m . As a matter of fact, in order to have $\omega \in N_m(\tau_j)$ we must have $\tau_j > \omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n}$, where $\omega x_{m+1}^{d_{m+1}} \cdots x_n^{d_n} \in N$, then the exponent of x_m will be the first checked in the lexicographic test and so it will be limited by d_m .

According to the values of this exponent, the ones associated to smaller variables will vary.

Remark 5.6. At the beginning of the algorithm, we imposed the monomials τ_j ,

$j = 1, \dots, r$ to be in increasing order with respect $<$. The steps made by the algorithm on each τ_j are totally independent both on those made and on those to be made on a monomial τ_k (it is indifferent whether $j \geq k$) belonging to G , so we will obtain the same factorizations even if we launch the computation on a list of unordered monomials.

Clearly, the result of our computation is not the reduced Groebner basis of the given ideal, it is only one of the minimal Groebner bases but we can obtain the reduced Groebner basis via simple reduction.

We decided to put the monomials in such an order because we want *every polynomial to be reduced with respect to the ‘previous’ ones*.

If f_j is one of our resulting polynomials and $Lt(f_j) = \tau_j$, the polynomials utilizable to reduce f_j (the previous ones) must be necessarily all and only the ones having as leading terms elements in G lower than the given τ_j .

The algorithm terminates because it works on:

1. points in the finite set \mathbf{X} ;
2. monomials $\tau \in G$ (they are in a finite number, [16]);
3. a finite set of variables.

Let us study the correctness of the algorithm.

Lemma 5.7. *The factorized polynomials obtained from our algorithm vanish on all the points of the set \mathbf{X} .*

Proof: Suppose we want to construct γ_τ with $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $\mu = x_1^{\beta_1} \cdots x_n^{\beta_n}$, corresponding to a point $P_\mu \in \mathbf{X}$ through Cerlienco-Mureddu Correspondence.

Let $\mu < \tau$, then at least one of the exponents of the variables appearing in μ is lower than the corresponding in τ , say $\beta_i < \alpha_i$, so μ is linked to an element of $N_i(\tau)$ and so it can, alternatively:

- belong to $A_{i\delta}(\tau)$ for some δ ;
- be such that the corresponding point already annihilates the polynomial found.

If $\mu > \tau$ (since $\tau \notin N$, it is surely impossible that $\tau = \mu$) then there will be a point $P_{\mu'}$ such that

$$\pi_j(P_\mu) = \pi_j(P_{\mu'}),$$

corresponding to a $\mu' < \tau$.

We then use μ' and we come back to the previous situation. \diamond

Corollary 5.8. *The ideal generated by these polynomials is exactly $I(\mathbf{X})$.*

Proof: By the previous lemma, the polynomials vanish on all the points of the set \mathbf{X} and the equality comes out by reasons of multiplicity \diamond

The resulting polynomials form a minimal Groebner basis because:

- they vanish on all the points of \mathbf{X} ;
- their heads form exactly $G(I(\mathbf{X}))$.

Notice that we can obtain the current interpolating polynomial applying Moeller algorithm to the projection through π_m of the points of the current $A_{m\delta}(\tau)$ ([14]).

Example 5.9. *Let*

$\mathbf{X} := \{(4, 0, 0), (2, 1, 4), (2, 4, 0), (3, 0, 1), (2, 1, 3), (1, 3, 4), (2, 4, 3), (2, 4, 2), (1, 0, 2)\}$.

$P_1 := (4, 0, 0) : \text{it is a single point, so } \Phi(\{(4, 0, 0)\}) = (0, 0, 0)$

$P_2 := (2, 1, 4) : s = 1, m = 1, (1, 0, 0)$

$P_3 := (2, 4, 0) : s = 2, m = 2, (0, 1, 0)$

$P_4 := (3, 0, 1) : s = 1, m = 1, (2, 0, 0)$

$P_5 := (2, 1, 3) : s = 3, m = 2, (0, 0, 1)$

Remark that $\gamma_{2\tau}$ is actually $\gamma_{21\tau}$.

$$\tau = x_2^2$$

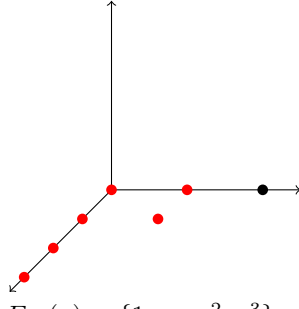
$$N_1(\tau) = \emptyset;$$

$$A_1(\tau) = \emptyset;$$

$$B_1(\tau) = \emptyset$$

$$m = 2:$$

$$D_{20}(\tau) = \mathbf{X}$$



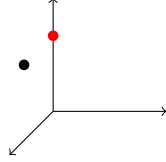
$$E_{22}(\tau) = \{1, x_1, x_1^2, x_1^3\};$$

$$\gamma_{22\tau} = 2x_2 - x_1^2 + 7x_1 - 12;$$

$$\xi_{22} = (x_2 - 4x_1 + 4)(2x_2 - x_1^2 + 7x_1 - 12)$$

$$D_{22}(\tau) = \emptyset;$$

$$\tau = x_1 x_3$$



$$m = 2:$$

$$N_2(\tau) = \{1\}.$$

$$D_{20}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\}$$

$$\delta = 1:$$

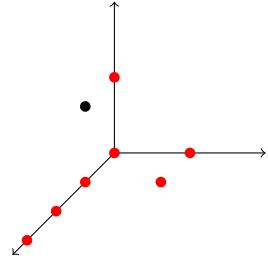
$$D_{21}(\tau) = D_{20}(\tau);$$

$$m = 3:$$

$$N_3(\tau) = \{1, x_1, x_2, x_1^2, x_3, x_1^3, x_1 x_2\};$$

$$\zeta_{m\tau} = (x_1 - 2);$$

$$D_{30}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\};$$



$$E_{31}(\tau) = \{1, x_1, x_1^2, x_2\};$$

$$N_2(\tau) = \{1, x_1, x_1^2, x_1^3, x_2, x_1 x_2\}; \delta = 1:$$

$$A_{21}(\tau) = \{(2, 4, 0), (1, 0, 2)\};$$

$$E_{21}(\tau) = \{1, x_1\};$$

$$\gamma_{21\tau} = x_2 - 4x_1 + 4$$

$$\xi_{21} = \gamma_{1\tau} \gamma_{21\tau} = x_2 - 4x_1 + 4;$$

$$D_{21}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1),$$

$$(2, 1, 3), (1, 3, 4)\};$$

$$\delta = 2:$$

$$A_{22}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (1, 3, 4)\}$$

The terms $vx_m^{d_m-\delta}$ are $1, x_1, x_1^2, x_1^3$ and they correspond exactly to P_1, P_2, P_4, P_6 .

$$N_1(\tau) = \{1\};$$

$$A_1(\tau) = \{(2, 1, 3)\};$$

$$B_1(\tau) = \{2\}$$

$$\gamma_{1\tau} = (x_1 - 2)$$

$$\delta = 1:$$

$$A_{31}(\tau) = \{(4, 0, 0), (3, 0, 1), (1, 3, 4), (1, 0, 2)\}$$

The terms are $1, x_1, x_1^2, x_1^3, x_2, x_1 x_2$, corresponding to $P_1, P_2, P_3, P_4, P_6, P_9$, but we can neglect P_2, P_3 .

$$\begin{aligned}\gamma_{31}(\tau) &= 6x_3 - 4x_2 + x_1^2 - x_1 - 12; \\ \xi_{31} &= (x_1 - 2)(6x_3 - 4x_2 + x_1^2 - x_1 - 12); \\ D_{31}(\tau) &= \emptyset.\end{aligned}$$

The desired polynomial is $\gamma_{3\tau} = \gamma_{31}(\tau)$.

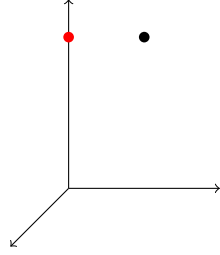
$$\tau = x_2 x_3^2$$

$$N_1(\tau) = \emptyset;$$

$$A_1(\tau) = \emptyset;$$

$$B_1(\tau) = \emptyset$$

$$m = 2:$$



$$N_2(\tau) = \{1\};$$

$$D_{20}(\tau) = \mathbf{X};$$

$$\delta = 1:$$

$$A_{21}(\tau) = \{(2, 4, 2)\};$$

$$E_{21}(\tau) = \{1\};$$

$$\gamma_{21\tau} = x_2 - 4$$

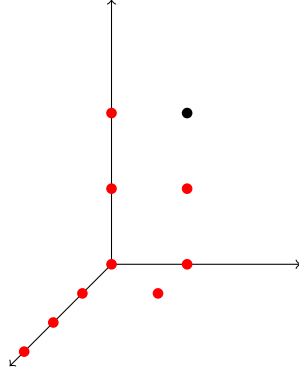
$$\xi_{21} = x_2 - 4;$$

$$D_{21}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (2, 1, 3), (1, 3, 4), (1, 0, 2)\};$$

$$m = 3:$$

$$\zeta_{3\tau} = x_2 - 4$$

$$D_{30}(\tau) = D_{21}(\tau);$$



$$N_3(\tau) = N(\mathbf{X});$$

$$\delta = 1:$$

$$A_{31}(\tau) = \{(2, 1, 3)\}.$$

$$E_{31}(\tau) = \{1\};$$

$$\gamma_{21\tau} = x_3 - 3$$

$$\xi_{31} = (x_2 - 4)(x_3 - 3);$$

$$D_{31}(\tau) = \{(4, 0, 0), (2, 1, 4), (3, 0, 1), (1, 3, 4), (1, 0, 2)\};$$

$$\delta = 2:$$

$$A_{32}(\tau) = D_{31}(\tau);$$

$$E_{32}(\tau) = \{1, x_1, x_1^2, x_1^3, x_2\};$$

$$\gamma_{32\tau} = x_3 - 4x_2 - 5x_1^3 + 41x_1^2 - 96x_1 + 48;$$

$$\xi_{32} = (x_2 - 4)(x_3 - 3)(x_3 - 4x_2 - 5x_1^3 + 41x_1^2 - 96x_1 + 48);$$

$$D_{32}(\tau) = \emptyset;$$

$$\gamma_{3\tau} = (x_3 - 3)(x_3 - 4x_2 - 5x_1^3 + 41x_1^2 - 96x_1 + 48);$$

$$\tau = x_3^3$$

$$N_1(\tau) = \emptyset;$$

$$A_1(\tau) = \emptyset;$$

$$B_1(\tau) = \emptyset$$

$$m = 2:$$

$$D_{20}(\tau) = \mathbf{X};$$

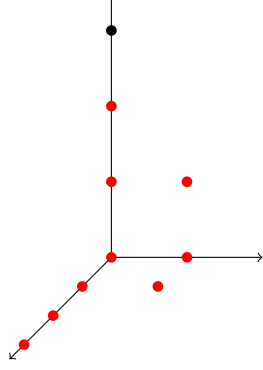
$$N_2(\tau) = \emptyset;$$

$$\delta = 1:$$

$$A_{21}(\tau) = \emptyset;$$

$$D_{21}(\tau) = \mathbf{X};$$

$m = 3$:
 $D_{30} = \mathbf{X}$;



$$\begin{aligned} N_3(\tau) &= N(\mathbf{X}); \delta = 1: \\ A_{31}(\tau) &= \{(2, 4, 2)\}; \\ E_{31}(\tau) &= \{1\}; \\ \gamma_{31\tau} &= x_3 - 2; \\ \xi_{31} &= x_3 - 2; \\ D_{31}(\tau) &= \{(4, 0, 0), (2, 1, 4), (2, 4, 0), \\ &\quad (3, 0, 1), (2, 1, 3), (1, 3, 4), (2, 4, 3)\}; \end{aligned}$$

$\delta = 2$:

$$A_{32}(\tau) = \{(2, 1, 3), (2, 4, 3)\};$$

$$E_{32}(\tau) = \{1, x_2\};$$

$$\gamma_{32\tau} = x_3 - 3;$$

$$\xi_{32} = (x_3 - 2)(x_3 - 3);$$

$$D_{32} = \{(4, 0, 0), (2, 1, 4), (2, 4, 0), (3, 0, 1), (1, 3, 4)\};$$

$\delta = 3$:

$$A_{33}(\tau) = D_{32};$$

$$E_{33}(\tau) = \{1, x_1, x_1^2, x_1^3, x_2\};$$

$$\gamma_{33\tau} = 6x_3 + 8x_2 - 5x_1^3 + 35x_1^2 - 54x_1 + 24;$$

$$\xi_{33} = (x_3 - 2)(x_3 - 3)(6x_3 + 8x_2 - 5x_1^3 + 35x_1^2 - 54x_1 + 24);$$

$$D_{33}(\tau) = \emptyset;$$

The required polynomial is $\gamma_{3\tau} = (x_3 - 2)(x_3 - 3)(6x_3 + 8x_2 - 5x_1^3 + 35x_1^2 - 54x_1 + 24)$.

Then our minimal Groebner basis of the ideal associated to \mathbf{X} with respect to the given order is:

$$\begin{aligned} \mathcal{G}(I(\mathbf{X})) = \Big\{ & x_1^4 - 10x_1^3 + 35x_1^2 - 50x_1 + 24, x_2x_1^2 - 3x_2x_1 + 2x_2, \\ & x_2^2 - 2x_2x_1 - x_2 + 2x_3 - 16x_1^2 + 38x_1 - 24, x_3x - 2x_3 - \frac{2}{3}x_2x_1 + \frac{4}{3}x_2 + \\ & + \frac{1}{6}x^3 - \frac{1}{2}x_1^2 - \frac{5}{3}x_1 + 4, x_3^2x_2 - 4x_3^2 - 7x_3x_2 + 28x_3 + \frac{8}{3}x_2x_1 + \\ & + \frac{20}{3}x_2 - \frac{16}{3}x^3 + 48x^2 - \frac{344}{3}x_1 + 32, x_3^3 - 5x_3^2 + \frac{8}{3}x_3x_2 - \frac{14}{3}x_3 - \frac{16}{9}x_2x_1 \\ & - \frac{40}{9}x_2 + \frac{73}{9}x_1^3 - \frac{197}{3}x_1^2 + \frac{1358}{9}x_1 - 72 \Big\}, \end{aligned}$$

obtained by our polynomials by the reductions stated in the Axis of Evil Theorem.

Finally, we remark that:

1. let $\tau_j = x_1^{d_1} \cdots x_n^{d_n} \in G$. The polynomial we are looking for must contain exactly $\sum_{i=1}^n d_i$ factors. It is impossible that the algorithm stops before, so it is impossible that a partial product vanishes on the whole \mathbf{X} . In fact, if so, there would be a polynomial $f \in I$ such that $T(f) \notin (G)$ (we know the minimal basis G before starting the Axis of Evil process);
2. if we obtain a factorized polynomial f such that its leading term $T(f)$ belongs to the minimal basis G , then f vanishes over all \mathbf{X} , because of 5.7.

Example 5.10. Consider the following ideal, given with its primary decomposition:

$$J := (x_1^2, x_2 + x_1, x_3) \cap (x_1^2, x_2 - x_1, x_3 - 1) = \\ = (x_1^2, x_1x_2, x_2^2, x_1x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2, x_2x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2, x_3^2 - x_3) \triangleleft \mathbb{C}[x_1, x_2, x_3].$$

Call its generators f_1, \dots, f_6 , considering them in the correct order.

It is 0-dimensional because $x_1^2, x_2^2, x_3^2 \in \text{In}(J)$ (see [16]), but it is not radical: its radical is $\sqrt{J} = (x_2, x_3^2 - x_3, x_1)$.

For such an ideal the Axis of Evil does not hold.

Consider the polynomial $f_4 = x_1x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2$.

By the Axis of Evil theorem (5.1), its factorization should be of the form:

$$(x_1 + \dots)(x_3 + \dots)$$

and we should have

$$x_1x_3 - \frac{1}{2}x_1 - \frac{1}{2}x_2 + Px_1^2 + Qx_1x_2 + Rx_2^2, \quad P, Q, R \in \mathbb{C}[x_1, x_2, x_3],$$

since we can only reduce deleting the multiples of x_1^2, x_1x_2, x_2^2 , in order to obtain f_4 . In order to have the correct product we must have $-\frac{1}{2}x_2$ in it. We can not obtain it through reductions, so the only chance is that we have a product of the form

$$k * hx_2,$$

with h, k constants such that $hk = -\frac{1}{2}$, in particular both different from 0.

A priori, we can have two possibilities:

- $(x_1 + k + \dots)(x_3 + hx_2 + \dots);$
- $(x_1 + hx_2 + \dots)(x_3 + k + \dots).$

The second one is impossible: the polynomial having x_1 as head can not contain variables greater than x_1 , so we consider only:

$$(x_1 + k + \dots)(x_3 + hx_2 + \dots).$$

We will then obtain

$$x_1x_3 + hx_1x_2 + kx_3 - \frac{1}{2}x_2 + \dots$$

We can delete the term x_1x_2 but it remains kx_3 which can not be reduced.

6 Corollaries.

We enumerate here some famous theorems which can be easily proved as corollaries of the Axis of Evil Theorem. For more details see, for example, [16].

Here we provide the general statements of these results, but clearly they can only be deduced under the hypothesis of the Axis of Evil theorem

The first one is *Lazard Structural Theorem*, which describes the structure of a minimal lexicographical Groebner basis of an $I \triangleleft k[x_1, x_2]$.

The original proof considers $P = k[x_1, x_2] = k[x_1][x_2]$ and it is based on the fact that $k[x_1]$ is a Principal Ideal Domain (PID).

Norton-Sălăgean [17] reformulated it using, more generally, $R[x]$ with R PIR.

We briefly recall the following

Definition 6.1. The content $r_f \in R$, with R PIR, of a polynomial $f(x) \in R[x]$ is the GCD of its coefficients. A polynomial $f(x) \in R[x]$ is called primitive if $r_f = 1$.

The primitive part of $f(x) \in R[x]$ is the polynomial $p_0(x) \in R[x]$ such that

$$f(x) = r_f p_0(x).$$

Let R be a PIR, $P := R[x]$. Let $I \triangleleft P$ e $F := \{f_0, \dots, f_s\}$ a minimal Groebner basis of I ordered in such a way that, called $d(i) := \deg(f_i)$, $\forall i, 0 \leq i \leq s$

$$d(0) \leq \dots \leq d(s).$$

Define then $c_i = lc(f_i)$, $r_i \in R \setminus \{0\}$ e $p_i \in P$ the leading coefficient, the content and the primitive part of f_i , for all $1 \leq i \leq n$.

Theorem 6.2 (Lazard). *If, moreover, R is a PID, then:*

- $f_0 = PG_1 \cdots G_{s+1}$;
- $f_j = PH_j G_{j+1} \cdots G_{s+1}$, $1 \leq j \leq s$.

where

1. $d(1) < \dots < d(s)$;
2. $G_i \in R$, $1 \leq i \leq s+1$ is such that $c_{i-1} = G_i c_i$
3. $P = p_0$ (the primitive part of $f_0 \in R[x]$);
4. $H_i \in R[x]$ is a monic polynomial of degree $d(i)$ in x , for all i ;
5. for all i we have $H_{i+1} \in (G_1 \cdots G_i, H_1 G_2 \cdots G_i, \dots, H_{i-1} G_i, H_i)$;
6. $r_i = G_{i+1} \cdots G_s$

Theorem 6.3 (Norton-Sălăgean). *With the previous notation, each*

$$p_i \in (f_j, j < i) : r_i.$$

In fact, we have $r_i = \prod_{m=1}^{n-1} \prod_{\delta=1}^{d_m} \gamma_{m\delta t_i}$ and $p_i = \prod_{\delta=1}^{d_n} \gamma_{n\delta t_i}$.

The second well-known result which can be straightforwardly derived from the Axis of Evil Theorem is the well known *Elimination Theorem* (see [2] for details)

Theorem 6.4 ([19]). *Let $I \triangleleft k[x_1, \dots, x_n]$ an ideal, take the lexicographical ordering induced by $x_1 < \dots < x_n$ and call I_j the j -th elimination ideal $I_j = I \cap k[x_1, \dots, x_j]$. Let \mathcal{G} be a Groebner basis of I , then $\mathcal{G}_j = \mathcal{G} \cap k[x_1, \dots, x_j]$ is a Groebner basis of I_j .*

The following result, *Kalkbrener theorem* ([13], [16]), is another consequence of the Axis of Evil Theorem and it is a stronger characterization of the lexicographical ordering.

For each subset $L \subset k[x_1, \dots, x_n]$, $i = 1, \dots, n$, $\forall \delta \in \mathbb{N}$ set

$$L_{i\delta} = \{p \in L, |p \in k[x_1, \dots, x_i], \deg_i(p) \leq \delta\}$$

and

$$Lp_{i,\delta} = \{Lp(p), p \in L_{i,\delta}\}.$$

Theorem 6.5 (Kalkbrenner). *With the previous notations, considered an ideal $I \triangleleft k[x_1, \dots, x_n]$ and a Groebner basis \mathcal{G} of it, these forms are equivalent:*

- \mathcal{G} is a Groebner basis of I w.r.t, the lexicographical order $<$ induced by $x_1 < \dots < x_n$;
- $Lp_{i,\delta}(\mathcal{G})$ is a Groebner basis of $Lp_{i,\delta}(I)$, $i = 1, \dots, n$, $\forall \delta \in \mathbb{N}$.

Let us now mention *Gianni-Kalkbrenner theorem*, whose situation is a bit more complicated (see [12], [7], [16]).

Theorem 6.6 (Gianni-Kalkbrenner). *Let $I \triangleleft k[x_1, \dots, x_n]$ an ideal and \mathcal{G} w.r.t the lexicographical order $<$ induced by $x_1 < \dots < x_n$. As before we define also $\mathcal{G}_d = \mathcal{G} \cap k[x_1, \dots, x_d]$.*

Consider $\alpha = (b_1, \dots, b_d) \in V(I_d)$ and define the projection map

$$\Phi_\alpha : k[x_1, \dots, x_n] \rightarrow k[x_{d+1}, \dots, x_n]$$

$$f(x_1, \dots, x_n) \mapsto f(b_1, \dots, b_d, x_{d+1}, \dots, x_n).$$

Let σ be the minimal value such that $\Phi_\alpha(Lp(g_\sigma)) \neq 0$ and j, δ the values such that

$$g_\sigma = Lp(g_\sigma)x_j^{\delta+1} + \dots \in k[x_1, \dots, x_j] \setminus k[x_1, \dots, x_{j-1}].$$

Then

1. $j = \delta + 1$
2. $\forall g \in \mathcal{G}_d, \Phi_\alpha(g) = 0$;
3. $\forall g \in \mathcal{G}_{d+\delta}, \Phi_\alpha(g) = 0$;
4. $\Phi_\alpha(g_\sigma) = \gcd(\Phi_\alpha(g), g \in \mathcal{G}_{d+1}) \in k[x_{d+1}]$;
5. $\forall b \in k, (b_1, \dots, b_d, b) \in V(I_{d+1}) \Leftrightarrow \Phi_\alpha(g_\sigma)(b) = 0$.

Clearly (1 – 3) are essentially a corollary of theorem 6.3; on the other side, (4 – 5) apparently cannot be deduced from the Axis of Evil Theorem.

7 Acknowledgement.

I wish to thank M. G. Marinari for her help, ideas and suggestions while studying this subject.

References

- [1] M.E. Alonso, M.G. Marinari, T. Mora, *The big Mother of all Dualities 2: Macaulay Bases*, Applicable Algebra in Engineering, Communication and Computing archive Vol. 17 Issue 6, November 2006, 409 – 451.
- [2] Buchberger B., *Gröbner Bases: An Algorithmic Method in Polynomial Ideal Theory*, in Bose N.K. (Ed.) *Multidimensional Systems Theory* (1985), 184–232, Reider

- [3] L. Cerlienco, M. Mureddu, *Algoritmi combinatori per l'interpolazione polinomiale in dimensione ≥ 2* , preprint(1990).
- [4] L. Cerlienco, M. Mureddu, *From algebraic sets to monomial linear bases by means of combinatorial algorithms*, Discrete Math. 139, 73 – 87.
- [5] L. Cerlienco, M. Mureddu, *Multivariate Interpolation and Standard Bases for Macaulay Modules*, J. Algebra 251 (2002), 686 – 726.
- [6] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann: SINGULAR 3-1-4 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de> (2012).
- [7] Gianni P., *Properties of Gröbner Bases under Specialization*, L. N. Comp. Sci. **378** (1987), 293–297, Springer
- [8] D. Lazard, *Ideal Basis and Primary Decomposition: Case of two variables*, J. Symb. Comp. 1 (1985), 261 – 270.
- [9] M.G. Marinari and Teo Mora, *Cerlienco-Mureddu Correspondence and Lazard Structural Theorem.*, Revista Investigación Operacional, Vol.27, No.2, 155-178, 2006.
- [10] M.G. Marinari and Teo Mora, *A remark on a remark by Macaulay or Enhancing Lazard Structural Theorem.*, Bulletin of the Iranian Mathematical Society Vol. 29 No. 1 (2003), pagg. 1 – 45.
- [11] M.G. Marinari and Teo Mora, *Some Comments on Cerlienco-Mureddu Algorithm and Enhanced Lazard Structural Theorem*, Rejected by ISSAC-2004 (2004).
- [12] M. Kalkbrenner, *Solving Systems of Algebraic Equations by Using Groebner Bases*, L. N. Comp. Sci. 378 (1987), pagg. 282 – 292, Springer.
- [13] Kalkbrenner M., *On the stability of Gröbner Bases under specialization*, J. Symb. Comp. **24** (1997), 51–58
- [14] M.G. Marinari, H.M Moeller, T. Mora, *Groebner Bases of Ideals Defined by Functionals with an Application to Ideals of Projective Points*, Applicable Algebra in Engineering, Communication and Computing, vol. 4, 1993, Springer.
- [15] M.G. Marinari, L. Ramella *Borel Ideals in three variables*, Beiträge zur Algebra und Geometrie. Contributions to Algebra and Geometry, Vol 47 (2006), N. 1, 195 – 209.
- [16] T. Mora, *Solving polynomial equation systems: Macaulay's paradigm and Groebner technology*, Cambridge University Press, 2005.
- [17] G.H. Norton, A. Sălăgean, *Strong Gröbner bases for polynomials over a principal ideal ring*, Bull. Austral. Math. Soc. **64** (2001), 505–528
- [18] S. Steidel, *pointid.lib*. Procedures for computing a factorized lex GB of the vanishing ideal of a set of points via the Axis-of-Evil Theorem (M.G. Marinari, T. Mora) (2011).
- [19] Trinks W., *Über B. Buchberger Verfahren, Systeme algebraischer Gleichungen zu lösen*, J. Numb. Th. **10** (1978), 475–488